PHASE PORTRAIT ANALYSIS FOR NAVIER–STOKES EQUATIONS IN A STRIP WITH OMITTED PRESSURE TERM

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ABSTRACT
In this paper, we study Navier – Stokes Equations in an infinite strip \( \Omega = \mathbb{R} \times [-1,1] \). We used the Uniformly Local Spaces as with focus on model situation where pressure has been omitted in the momentum equation. We obtain a priori estimates for the solutions and use the phase – portrait analysis to obtain bounds for them. We caution that the method used in this paper cannot be applied in general situation but is peculiar to only circumstances where the pressure term has been artificially removed.

Keywords: Phase-Portrait, Uniformly local Spaces, Weighted Energy Estimates

INTRODUCTION
In this paper we seek to find the weighted energy estimate, in uniformly local spaces, of the Navier – Stokes Equations in an infinite strip by omitting the pressure term. The phase portrait method has been used to study the solutions and obtain bounds for them.

The Navier Stokes Systems:
\[
\begin{align*}
\partial_t u + (u, \nabla) - \Delta u + \nabla p &= f \\
\text{div} u &= 0, \quad u|_{\partial \Omega} = 0, \quad u|_{t=0} = u_0
\end{align*}
\]
(1.1)
is considered in \( \Omega = \mathbb{R} \times [-1,1] \).

The global in time estimate for 2D Navier – Stokes equations was first obtained for bounded domains in the works of Ladyzhenkiya(1972). Later on, the unbounded domain case was treated by Abergel(1979) and Babin(1992), and the forcing term was required to lie in some weighted space. However, the dimension estimate of the attractor for more general forces. We start with the class of admissible weight functions.

Let us briefly state the definition and basic properties of weight functions and weighted functional spaces as presented by Zelik(2007) which is too restrictive a decay condition. So under this choice of the phase space many hydrodynamical objects like Poiseuille flows (infinite energy), Kolmogorov flows etc. cannot be considered in the circumstances; because of the above restrictions hold on our model equation (1.1) we are unable to consider constant solutions space periodic solutions etc. which will hinder us from capturing physically relevant solutions.

PRELIMINARIES: UNIFORM AND WEIGHTED ENERGY SPACES
In this section, we introduce and briefly discuss the weighted and uniformly local spaces which are the main technical tools to deal with infinite-energy solutions, see Zelik (2007) for more detailed exposition. These tools will help us to obtain estimates for our equations (1.1-1.6) in unbounded domain \( \Omega = \mathbb{R} \times [-1,1] \). We explain the space as follows:

Let us define \( B^1_{x_0} = 1 \) - a unit rectangle centered at \( (x_0,0) \) represented as:
\[
B^1_{x_0} = \left(x_0 - 1, x_0 + 1\right) \times \{0\} \subset \mathbb{R}^2
\]
(2.1)

Let us briefly state the definition and basic properties of weight functions and weighted functional spaces as presented by Zelik (2003), Anthony and Zelik (2014), Zelik (2013) and the references therein. Which will be systematically used throughout this project (see also Efendiev and Zelik (2002) for more details). We start with the class of admissible weight functions.

Definition 2.1. A function \( \phi \in C^0_{loc}(\mathbb{R}) \) is weight function of exponential growth rate \( \mu > 0 \) if the following inequalities hold:
\[
\phi(x + y) \leq C_\phi \phi(x) e^{\mu |y|}, \phi(x) > 0,
\]
(2.2)
For \( x, y \in \Omega = \mathbb{R} \).
We now introduce a class of weighted Sobolev spaces in a regular unbounded domains associated with weights introduced above. We need only the case where \( \Omega = \mathbb{R} \times [-1,1] \) is a strip which obviously have regular boundary. One would like to ask why we need weighted Sobolev Spaces; recall that the uniformly local spaces encountered some deficiencies in that they are not differentiable when the supremum is involved but the weighted energy spaces resolve this problem.

**Definition 2.2.**

\[
L^p_{\phi}(\Omega) = \{ u \in L^p_{\text{loc}}(\Omega) \mid \| u \|_{L^p_{\phi}(\Omega)} = \left( \int_{\Omega} \phi(x) |u(x)|^p \, dx \right)^{1/p} < \infty \}
\]

And

\[
L^p_{\phi}(\Omega) = \{ u \in L^p_{\text{loc}}(\Omega) \mid \| u \|_{L^p_{\phi}(\Omega)} = \sup_{x \in \mathbb{R}} \| u \|_{L^p_{\phi}(B(x,y))} \}
\]

\( x,y \in \Omega = \mathbb{R} \times [-1,1] \)

The uniformly local space \( L^p_{\phi}(\Omega) \) consists of all functions \( u \in L^p_{\text{loc}}(\Omega) \) for which the following norm is finite

\[
\| u \|_{L^p_{\phi}(\Omega)} = \sup_{x \in \mathbb{R}} \| u \|_{L^p_{\phi}(B(x,y))} < \infty
\]

If \( u \in L^\infty \Rightarrow u \in L^2_{\phi} \) and

\[
\| u \|_{L^2_{\phi}} \leq C \| u \|_{L^\infty}.
\]

This is because all functions that are bounded in \( L^\infty \) are also bounded in \( L^2_{\phi} \) but the reverse is not true.

\[
\| u \|_{L^p_{\phi}(B(0,1))} \leq \| u \|_{L^p_{\phi}(B(0,1))} \leq \| u \|_{L^\infty(\Omega)}
\]

Similarly, the uniformly local Sobolev spaces \( H^s_{\phi}(\Omega) \) consist of all functions \( u \in H^s_{\text{loc}}(\Omega) \) for which the following norm is finite:

\[
H^s_{\phi}(\Omega) = \{ u \in H^s_{\text{loc}}(\Omega) \mid \| u \|_{H^s_{\phi}(\Omega)} = \sup_{x \in \mathbb{R}} \| u \|_{H^s_{\phi}(B(x,y))} \}
\]

Where \( H^s \) is the space of all distributions whose derivative up to order \( s \) is in \( L^2 \). The following Lemma establishes the relationship between the spaces \( L^2_{\phi} \) and \( L^2_{\phi} \).

**Lemma 2.3.** Let \( \phi \) be a weight function of exponential growth rate, where \( \phi(x_0) = \phi_0 \) \( (x-x_0) \), satisfying

\[
\int \phi^2 \, dx < \infty \text{ then the following inequalities hold}
\]

\[
\| u \|_{L^2_{\phi}} \leq C_1 \| u \|_{L^2_{\phi}} \cdot \int \phi^2 \, dx
\]

(2.3)

Where \( C_1 \) and \( C_2 \) depend only on \( C_\phi \) and \( \tau \).

**Proof.** If \( u \in L^2_B \) then \( \| u \|_{L^2(B(x_0))} < C \), \( x_0 \in \mathbb{R} \) is bounded uniformly and

\[
\leq \sum_{N=0}^{\infty} \| \phi \|_{L^2[N,N+1]} \| u \|_{L^2[N,N+1]}^2
\]

\[
\leq \sum_{N=0}^{\infty} \| u \|_{L^2[N,N+1]}^2 \| \phi \|_{L^2[N,N+1]}^2
\]

This follows from the integral criterion for convergence i.e. \( \sum_{N=0}^{\infty} \phi^2(N) \) converges if and only if \( \int_{-\infty}^{\infty} \phi^2(x) \, dx < \infty \).

The opposite side of the proof takes \( u \in \sup_{x \in \mathbb{R}} \| u \|_{L^2_{\phi}}^2 \) so we need to estimate the norm \( \| u \|_{L^2_{\phi}}^2 \) hence

\[
\min_{x \in B(0,1)} \phi(x) \geq c_0 \text{ and using a specific weight function of the form}
\]

\[
(x+y) \leq C e^{\| x \|} \phi(y)
\]

so that

\[
\phi(y) \geq C e^{\| x \|} \phi(0) \geq C \phi(0)
\]

Take \( x=y \) to get

\[
\phi(y) \geq C e^{-\| x \|} \phi(0) \geq C \phi(0)
\]

And hence,

\[
\int_{B(x_0)} \phi^2 \, dx \leq C_2 \| u(x) \|_{L^2_{\phi}}^2 \leq C_1 \| u \|_{L^2_{\phi}}^2
\]

Before we conclude this section, we introduce a special weight
function which would be essentially used in the sequel. Let
\[
\int_{\Omega} \phi^2(x)|u(x)|^2 \, dx = \sum_{N=0}^{\infty} \int_{N}^{N+1} \phi^2(x)|u(x)|^2 \, dx
\]
\[
\phi(x) = (1 + \varepsilon^2|x|^2)^{\frac{1}{2}}
\]
We see that
\[
\phi'(x) = -\frac{\varepsilon^2|x|}{(1 + \varepsilon^2|x|^2)^{\frac{3}{2}}}
\]
\[
\phi'(x) \leq -C \frac{\varepsilon^2|x|}{(1 + \varepsilon^2|x|^2)^{\frac{3}{2}}}
\]
\[
\phi'(x) \leq C\varepsilon\phi(x)
\]
This weight, in addition, to (2.5) (which holds for every positive \( \mu \)), satisfies the following property:
\[
\left| \phi'(x) \right| \leq C\phi(x) < C\phi(x)
\]
A bit more general are the weights \( \phi(x)^N, N \in \mathbb{R}, N \neq 0 \), which are also the weights of exponential growth rate \( \mu \) for any \( \mu > 0 \) and satisfy the analog of (2.9) where the exponent 2 is replaced by \( \frac{N+1}{N} \).

In the sequel, we will need the Sobolev embedding and interpolation inequalities for the case of weighted spaces with the embedding constants independent of \( \varepsilon \rightarrow 0 \). Following Babin and Vishik (1990); and Robinson (2001), such inequalities can be derived from the analogous non-weighted situation utilizing the isomorphism \( T_{\phi}u = \phi \cdot u \) between weighted and non-weighted spaces.

**Lemma 2.4.** Let \( \phi \) be a weight function defined by (2.7). Then, for all \( l \) and of exponential growth rate, the map \( T_{\phi} \) is an isomorphism between \( W^{1,p}(\Omega) \) and \( W^{1,p}_{\phi}(\Omega) \) and the following inequalities hold:
\[
C_{1}\|\phi u\|_{W^{1,p}_{\phi}} \leq \|u\|_{W^{1,p}(\Omega)} \leq C_{2}\|\phi u\|_{W^{1,p}_{\phi}}
\]
(2.10)

Where \( C_{1} \) and \( C_{2} \) are independent of \( \varepsilon \) but may depend on \( l \) and \( p \)

**Proof let us take** \( V = \phi u \), then for \( u \in L^2_{\phi} \) we have:
\[
\|\phi u\|^2_{L^2_{\phi}} = \int \phi^2(x)|u|^2 \, dx = \int |V|^2 \, dx = \|V\|^2_{L^2_{\phi}}
\]
(2.11)

(2.11) shows isometry between spaces \( L^2_{\phi} \) and \( L^2_{\phi} \). The same procedure give isomorphism between \( L^2_{\phi} \) and \( L^p \). We show that \( T_{\phi} \) is an isomorphism between \( W^{1,2} \) and \( W^{1,2}_{\phi} \). Let \( u \in W^{1,2}_{\phi}(\Omega) \).

Then,
\[
\|V\|^2_{W^{1,2}_{\phi}} = \int (|\nabla V|^2 + |V|^2) \, dx
\]
\[
\|\phi u\|^2_{W^{1,2}_{\phi}} = \int (|\nabla (\phi u)||^2 + |\phi u|^2) \, dx + \int |\phi u|^2 \, dx
\]
\[
= \int (|\nabla u|^2 + |\phi u|^2) \, dx + \int |\phi u|^2 \, dx
\]

By Young’s inequalities and the relation that \( \phi \leq \frac{1}{2} \phi \) (since only small values of \( \varepsilon \) are important for us, for convenience, we assume that \( \varepsilon \leq \frac{1}{2} \)), we have
\[
\|\phi u\|^2_{W^{1,2}_{\phi}} < 2 \int (|\nabla u|^2 + |\phi^2|u|^2) \, dx = C_{1}\|u\|^2_{W^{1,2}_{\phi}}
\]
(2.12)

This gives the left inequality of (2.10) (in the particular case \( l = 1, p = 2 \)). Let us prove now the right one:
\[
\|\phi u\|^2_{W^{1,2}_{\phi}} = \int \phi^2 |\nabla u|^2 \, dx + \int \phi^2 |u|^2 \, dx
\]
\[
\leq \int (|\nabla (\phi^2 u)|^2 + |\phi^2 u|^2) \, dx + \int |\phi^2|u|^2 \, dx + \int |\phi^2|u|^2 \, dx
\]
\[
= \int (|\nabla u|^2 + |\phi^2|u|^2) \, dx + \int |\phi^2|u|^2 \, dx
\]
By Cauchy Schwartz, Young’s inequalities and using the fact that \( \phi \leq \frac{1}{2} \phi \) we obtain
\[
\|\phi u\|^2_{W^{1,2}_{\phi}} < 2 \int (|\nabla u|^2 + \phi^2 |u|^2) \, dx = C_{1}\|u\|^2_{W^{1,2}_{\phi}}
\]
(2.13)

This gives the right inequality of (2.10). Thus, for the particular case \( l = 1, p = 2 \), (1.17) is proved. The proof in a general case is analogous and we leave it to the reader; we have the following relation
\[
\|\phi u\|^2_{W^{1,2}_{\phi}} < 2 \int (|\nabla u|^2 + \phi^2 |u|^2) \, dx = C_{1}\|u\|^2_{W^{1,2}_{\phi}}
\]
(2.14)

As required
Next corollary gives the weighted analogue of one interpolation inequality useful for what follows, see Triebel (1978).

**Corollary 2.5.** Let \( \phi \) be defined (2.7) and let \( u \in W^{1,2}_{\phi} \) then the following interpolation inequalities hold

\[
\|u\|_{L_{\phi}^3} \leq C \|\nabla u\|_{L_{\phi}^2} \|u\|_{L_{\phi}^4},
\]

(2.15)

Where the constant \( C \) is independent of \( \varepsilon \geq 0 \).

**Proof**

We know that for unweighted case, the embedding of \( H^1 \) into \( L^3 \) is performed by the following Sobolev embedding theorem:

\[
\frac{0}{2} \leq \frac{1}{3} \leq \frac{1}{6} \leq \frac{1}{2}
\]

(2.16)

Therefore \( H^{\frac{1}{3}} \rightarrow L^3 \); interpolating \( H^1 \) between \( L^3 \) and \( H^1 \) we have the following equation with exponent \( \theta \):

\[
\|V\|_{L_{\theta}^3}^3 \leq C \|\nabla V\|_{L_{\theta}^2} \|V\|_{L_{\theta}^4}^3
\]

(2.17)

For any \( V \in W^{1,2}_{\phi} \) where:

\[
\frac{1}{3} = 0 \times \theta + 1 \times (1 - 0) \Rightarrow \theta = \frac{1}{3}
\]

(2.18)

And also

\[
\|\nabla\|_{L_{\phi}^3} \leq C \|\nabla\|_{L_{\phi}^2} \|\|_{L_{\phi}^4}^3
\]

(2.19)

Taking \( V \in W^{1,2}_{\phi} \) in (1.26) and using Lemma 2.4, we obtain (2.14). This ends the proof.

Now, in order to obtain the proper estimates for the solutions in the uniformly local spaces, one can use the so-called weighted energy estimates as an intermediate step and utilize the relation

\[
\|u\|_{L_{\phi}^3} \sim \sup_{x \in \mathbb{R}} \|u\|_{L_{\phi}^2}^{\frac{3}{2}}
\]

Where \( \phi \) is a property chosen (square integral) weight function, 2.7.

**ESTIMATES FOR NAVIER – STOKES EQUATIONS WITHOUT PRESSURE**

We want to prove that any sufficiently regular solution of the Navier – Stokes proble; (1.1) in a cylinder satisfies the uniformly local estimate:

\[
\|u(t)\|_{L_{\phi}^3(\Omega)} \leq Q \left( \|u(0)\|_{L_{\phi}^3(\Omega)} + Q \left( \|f\|_{L_{\phi}^2(\Omega)} \right) \right)
\]

(3.1)

for some monotone function \( Q \). The first difficulty here is that, in contrast to the case of usual energy solutions, the function \( t \rightarrow \|u(t)\|_{L_{\phi}^3(\Omega)} \) is not differentiable due to the presence of supremum in the definition of \( L_{\phi}^3 \) - norm. This does not allow us to obtain estimate (3.1) directly by multiplying the equation by the appropriate factor and use Gronwall’s inequality. Instead, following the general strategy in, we deduce the weighted energy estimates as an intermediate step; multiplying the equation by \( \phi^2 u \) where \( \phi \) is a proper weight function described in equation 2.7. If we succeed to verify the analogue of (3.1) in all weighted spaces \( L_{\phi}^3(x_0, \Omega) \) uniformly with respect to all shifts \( x_0 \in \mathbb{R} \), the desired uniformly local estimate will obtained by taking the supremum with respect to \( x_0 \in \mathbb{R} \) and using Lemma 2.2. Thus, we need to multiply equation (1.1) by \( \phi^2 u \) where \( \phi = \phi(x) \) is an appropriate weight function in \( \chi_1 \) direction. But the nonlinear term will still remain unresolved since it will not disappear as in the bounded case. In fact it will produce a cubic nonlinearly \( \phi u^3 \). Note that the cubic term is not clear how to control the cubic term in order to produce an a priori estimate. Another setback is the fact that \( \phi^2 u \) is not divergence free so the pressure \( p \) does not disappear in the weighted energy equality and \( \phi \phi p, u \) will pose a problem closely related with finding a reasonable extension of the Helmholtz projector (to divergence free vector fields) to uniformly local spaces. In summary, we have at least two hurdles to overcome in order to close our estimates: to estimate the cubic term \( \phi u^3 \) produced by the nonlinear term and \( \phi \phi p, u \) when we multiply the momentum equation (1.1) by \( \phi^2 u \) and integrate over the domain \( \Omega \). Let us put this in perspective to have a clearer view of the terms when we multiply (1.1) by \( \phi^2 \chi(x_1) u \) (where \( \phi \chi(x_1) \) is defined as in (2.7) and \( \epsilon \) is a small positive constant to be determined later) and integrate over \( \Omega \) to obtain:

\[
\partial_t u, \phi^2 u + (u, V)u, \phi^2 u - (\nabla u, \phi^2 u) + (\Delta p, \phi^2 u) = (f, \phi^2 u)
\]

(3.2)

and hence,

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L_{\epsilon}^2}^2 + (u, V)u, \phi^2 u + v \|u\|_{L_{\epsilon}^2}^2 + (\Delta p, \phi^2 u) = (f, \phi^2 u) - v (u \phi \phi, V)
\]

(3.3)

Next, we try and resolve the nasty terms in (3.3) i.e. the second and fourth terms on the LHS; this is to help simplify them as much as possible. Using 2D coordinates; we now seek to estimate the non – linear term as follows:

\[
- \sum_{i,j} \int_{\Omega} u_j \partial_i u_i \phi^2 u_{i}dx
\]

\[
= - \sum_{i,j} \left( u_{i}u_{j} \partial_i u_i + u_{i}^2 \partial_i \phi^2 u_{i} \right) dx
\]

applying the divergent – free condition and collecting like terms we obtain

\[
2 \sum_{i,j} \int_{\Omega} u_i u_j \phi^2 \partial_i u_i dx = - \sum_{i,j} \int_{\Omega} u_i u_j \partial_i \phi^2 dx
\]

Simplifying the above sum integral with careful consideration that \( \phi \) is applied in the \( x_1 \) direction we obtain:

\[
- \sum_{i,j} \int_{\Omega} u_i u_j \phi \partial_i \phi^2 dx = - \int_{\Omega} \left( u_i^2 \partial_i u_i + u_{i}^2 \partial_i \phi^2 \right) dx
\]

(3.4)
Next, we have the pressure term to resolve but it simply not disappear because the equation:
\[
\int_\Omega \nabla p \cdot \phi u \, dx = -\int_\Omega p \nabla (\phi u) \, dx
\]
\[
= -\int_\Omega [\phi \nabla (u^2) + u \nabla \phi] \, dx
\]
\[
= 2(\phi \cdot p, u)
\]
But the application of the divergence free condition did not help exclude the term with pressure \((\phi \cdot p, u)\) - a difficult piece to estimate.

Just for the moment, we shall proceed with other terms of Navier – Stokes equation without the term containing the pressure and compute the estimate with the impression that we shall return later with good theory that will enable us overcome the difficulty posed by \((\phi \cdot p, u)\) and eventually close our estimate for Navier - Stokes equation in an unbounded domain \(\Omega \subset \mathbb{R}^2\).

Now, recall (3.3) using (3.4) and ignoring the term \((\phi \cdot p, u)\) we have:
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 + (\phi |\phi|, |u|^2) + v\|u\|^2_{I_\phi} + \|F \phi u\|_{I_\phi} = \langle f, \phi u \rangle - v \nabla |\nabla u| (3.5)
\]
The non – linear term is then estimated using corollary 2.1 and Poincare inequality to obtain the following:
\[
(\phi |\phi|, |u|^2) \leq (\epsilon \phi^3, |u|^3) \triangleq \epsilon \|u\|^2_{I_\phi} \leq C \epsilon \|u\|^2_{I_\phi}
\]
(3.6)
Let us tidy up these bits of the Navier – Stokes equation and write (3.5) to obtain:
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 - C \epsilon \|u\|^2_{I_\phi} \|\nabla u\|_{I_\phi} + v\|u\|^2_{I_\phi} + \|F \phi u\|_{I_\phi} = \langle f, \phi u \rangle - v \nabla |\nabla u| (3.7)
\]
By Cauchy Schwartz inequality on the RHS of (3.7) we obtain:
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 - C \epsilon \|u\|^2_{I_\phi} \|\nabla u\|_{I_\phi} + v\|u\|^2_{I_\phi} \leq ||\ell||_{I_\phi} \|u\|^2_{I_\phi}
\]
(3.8)
By Young's inequality on (3.8) we get
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 - C \epsilon \|u\|^4_{I_\phi} \frac{v\|u\|^2_{I_\phi}}{4} + v\|u\|^2_{I_\phi} + \frac{\|\ell\|_{I_\phi}^2}{2} + \frac{v\|u\|^2_{I_\phi}}{2} \geq \frac{\|\ell\|^2}{2} + \frac{v\|u\|^2_{I_\phi}}{2}
\]
(3.9)
This simplifies to:
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 - C \epsilon \|u\|^4_{I_\phi} \frac{v\|u\|^2_{I_\phi}}{4} + v\|u\|^2_{I_\phi} + \frac{\|\ell\|_{I_\phi}^2}{2} \geq \frac{\|\ell\|^2}{2} + \frac{v\|u\|^2_{I_\phi}}{2}
\]
(3.10)
Where \(\gamma\) is a positive constant to be determined later. Applying Poincare inequality on the second term of the RHS of (3.10) and assuming \(\epsilon < 1\) is small enough; the equation reduces to:
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 - C \epsilon \|u\|^4_{I_\phi} \frac{v\|u\|^2_{I_\phi}}{4} + v\|u\|^2_{I_\phi} + \frac{\|\ell\|_{I_\phi}^2}{2} \geq \frac{\|\ell\|^2}{2} + \frac{v\|u\|^2_{I_\phi}}{2}
\]
(3.11)
Take that \(\frac{\gamma}{2} \geq \frac{C_F}{2}\) for the purpose of achieving a positive linear term that will afford us not only global existence of solution but also dissipative.
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 - C \epsilon \|u\|^4_{I_\phi} \frac{v\|u\|^2_{I_\phi}}{4} + \frac{v\|u\|^2_{I_\phi}}{2} + \frac{\|\ell\|_{I_\phi}^2}{2} \geq \frac{\|\ell\|^2}{2} \geq \epsilon^{-1} C \|\ell\|^2_{I_\phi}
\]
(3.12)
Where estimate (2.1) from lemma 2.1 and \(\int \phi \xi^2(x) \, dx = \frac{C}{\gamma}\) are used in order to obtain the RHS of (3.12) and constant \(C\) and \(C_1\) are independent of \(\epsilon > 0\). We concisely get:
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{I_\phi}^2 - C \epsilon \|u\|^4_{I_\phi} \frac{v\|u\|^2_{I_\phi}}{4} \leq \frac{\|\ell\|^2}{2} \leq \epsilon^{-1} C \|\ell\|^2_{I_\phi}
\]
(3.13)
We take \(\|u\|^2_{I_\phi} = y(t)\) to obtain the following differential inequality:
\[
y'(t) + C\gamma y(t) \leq \epsilon^{-1} y_{\|u\|_{I_\phi}^2} + \frac{v\|u\|^2_{I_\phi}}{2}
\]
(3.14)
By change of variable \(\gamma = \frac{\gamma}{2}\) and upon using the fact that \(\|\ell(0)\|^2_{I_{\phi}} \leq \|\ell(0)\|^2_{I_{\phi}} \leq C\gamma \leq \|\ell(0)\|^2_{I_{\phi}} \leq \frac{v\|u\|^2_{I_\phi}}{2} \leq C\gamma\); we have, first that
\[
y(0) = \|u(0)\|^2_{I_\phi} \leq \frac{C}{\epsilon} \|u(0)\|^2_{I_\phi}
\]
(3.15)
Now, recall (3.14) and write it as: \(y'(t) + \gamma y(t) \leq \epsilon^{-1} y_{\|u\|_{I_\phi}^2} + \frac{v\|u\|^2_{I_\phi}}{2}\) so that for the initial data \(y(0) \leq C\epsilon^{-1}||u(0)||^2_{I_\phi}\) and by the above scaling, with its initial condition \(\gamma = C\epsilon \|u(0)\|^2_{I_\phi}\).

We shall seek to solve (3.14) to prove that it has global bounds for solutions because of the positive linear term on the LHS of (3.14). we state a lemma involving a two stage proof of the estimate for (3.14): The first part of the proof considers the case of a simple ODE, while the next considers a system of ODE:

**Theorem 4.1** Let \(y(t) = y(\gamma t)\) be absolutely continuous and satisfy for every small \(\epsilon\) the following differential inequality:
\[
y'(t) + \gamma y(t) \leq \epsilon^{-1} C_f \frac{y(t)}{2}
\]
(4.1)
\[
y(0) \leq \epsilon^{-1} C_0\]
(4.2)
For some \(C_f, C_0\) independent of \(\epsilon\). Then \(y(t)\) is globally bounded for all \(t \geq 0\):
\[
y(t) \leq \epsilon^{-1} (C_0 + \frac{C_f}{\gamma})\]
(4.3)
If \(\epsilon\) is small enough:
\[
\epsilon \leq C (C_f + C_0)
\]
(4.4)
And constant \(C\) is independent of \(\epsilon \to 0\).

**Proof:**
This portion of the proof will use the phase-potrait technique. Given
\[
z'(t) + \gamma z(t) \leq C_f + \epsilon z^2(t), \quad z(0) = \epsilon y(0)
\]
\[
\leq C_0\]
(4.5)
Let \(z(t) \leq w(t)\) so that the inequality part of (4.5) is exactly \(w(t)\):
\[
w' + \gamma w(t) = C_f + \epsilon w^2(t), \quad w(0) = z(0)\]
(4.6)
Bearing in mind that all solutions (4.5) lies below (4.6) with the same initial condition, we conclude that \(z(t) \leq w(t)\) for all \(t\). So it is enough to estimate \(w(t)\).

To study the dynamics of ODE (4.6), we find the following equilibrium solutions:
\[
w_x(\epsilon) = \frac{2C_f}{(\gamma + \sqrt{\gamma^2 - 4C_f \epsilon})} = \frac{C_f}{\gamma} - \frac{2C_f \epsilon}{\gamma^2} + \ldots = \frac{C_f}{\gamma}
\]
\[ w_2(\epsilon) = \frac{\sqrt{\epsilon^2 - 4\epsilon C_f}}{2\epsilon} = \frac{\epsilon}{\epsilon} - \frac{C_f}{\epsilon^2} \approx \frac{2\epsilon}{\epsilon} \]

And from \( w_2(\epsilon) \) we see that \( \lim_{\epsilon \to 0} \frac{(\sqrt{\epsilon^2 - 4\epsilon C_f})}{2\epsilon} = \frac{C_f}{\epsilon} \). We see that \( w_1(\epsilon) \) remains bounded as \( \epsilon \to 0 \) and \( w_2(\epsilon) \approx \frac{C_f}{\epsilon} \) however, is exactly the root of the limit linear equation:

\[ w_2(\epsilon) \to \infty \text{ as } \epsilon \to 0. \]

Looking at the phase-portrait of the ODE, we get full description about the dynamics: There is a stable equilibrium near \( \epsilon = \frac{C_f}{\epsilon} \)

and all solutions around it which are less that \( w_2 \) are attracted to it. We know that \( w(t) \) will decay for all time tending to \( w_1(\epsilon) \) if \( w(0) \in [w_1(\epsilon), w_2(\epsilon)] \). Thus in that case we simply have \( z(t) \leq w(0) \leq z(0) \). The above choice of the initial condition is possible if \( \epsilon \) is such that \( \epsilon^2 - 4\epsilon C_f > 0 \), otherwise \( w(t) \) will blow up in finite time, we also exclude the choice of \( \epsilon \) that considers \( z(0) > w_2(\epsilon) \).

But in the case of \( z(0) < w_2(\epsilon) \) the solution will grow tending to \( w_1(\epsilon) \) as \( t \to \infty \). So in that case, \( z(t) < w_1(\epsilon) \). Finally, in both cases, \( z(t) \leq z(0) + w_1(\epsilon) \) and since \( w_1(\epsilon) \leq \frac{2C_f}{\epsilon} \) we obtain \( y(t) \leq \epsilon^{-1} z(t) \leq \epsilon^{-1} (C_0 + C_f) \).

We have shown how to derive the desired a priori estimate for the NSE in uniformly local spaces in the model situation where the pressure term is not taken into account. The phase portrait analysis has been used to get bounds for the solutions. Existence, uniqueness and further regularity of solutions may be obtained in a standard way by the Galerkin approximation. We must emphasize here, however, that this method applies only to model situation where pressure is formally omitted.

REFERENCES


