Science World Journal Vol 12(No 4) 2017
www.scienceworldjournal.org
ISSN 1597-6343

Full Length Research Article

SOME GENERALIZED TWO-STEP BLOCK HYBRID NUMEROV METHOD FOR SOLVING GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITHOUT PREDICTORS

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ABSTRACT
This paper proposes a class of generalized two-step Numerov methods, a block hybrid type for the direct solution of general second order ordinary differential equations. Both the main method and additional methods were derived via interpolation and collocation procedures. The basic properties of zero stability, consistency and convergence of these methods were also investigated. Results from numerical examples show significant improvement.

Keywords: Block Hybrid Methods, Ordinary Differential Equations, Continuous formulation, Interpolation and Collocation, Numerov method

INTRODUCTION

Consider the general second order ordinary differential equation of the form:

$$y'' = f(x, y, y') \tag{1.1}$$

with initial conditions

$$y(a) = y_a, y'(a) = y_0 \tag{1.2}$$

In particular, (1.1) arise in many physical phenomena such as electrical circuit, spring and buoyancy problems. Many of these problems may not be easily solved analytically; therefore the development of numerical method(s) becomes inevitable. Second order ordinary differential equations have attracted considerable attention and its theoretical and numerical studies have appeared in relevant literature.

The popular approach for solving (1.1) is by converting the problem to a system of first order ODE and then solving it using numerical method like the $R-K$ method and Linear multistep methods (Lambert, 1973 and 1991; Fatunla, 1988 and 1991). The major drawback inherent in this approach has been highlighted by Jator [6 and 7], Mehrkanoon [12], Awoyemi and Idowu [1] to include: Complicated computational work and lengthy execution time.

The studies on direct approach to higher order ODEs demonstrated the advantages in speed and accuracy. Some attention has been focused on direct solution of (1.1), Fatunla [1991] suggested the zero-stable 2-point block method to solve special second order ODE's in which the first derivative is absent. Omar *et al*. [13], Malid and Suleiman [10] studied parallel implementation of the direct block methods. The following scholars also study the direct numerical solution of (1.1), Onumanyi *et al*. [14]; Ismail *et al*. [5].

In this paper, a direct numerical solution to the general second order ODE's of the form (1.1) is proposed without recourse to the conventional way of reducing it to a system of first order ODE (Chan *et al*., 2004). We developed generalized schemes of continuous linear multistep methods (CLMS) of hybrid type via multistep collocation technique. This method helps to provide a continuous numerical scheme which accommodates all hybrid points, so that on substitution of an off-grid point, a hybrid scheme is obtained.

Derivation of the Block Hybrid Method

The hybrid collocation method that produces approximations y_{n+k}, y'_{n+k} to the general second order ODEs is given in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} + \alpha_q y_{n+q} = h^2 \sum_{j=0}^{s-1} \beta_j f_{n+j} \tag{2.1}$$

and the continuous formulation of (2.1) given in the form:

$$y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j} \tag{2.2}$$

In order to obtain (2.2), we approximate the solution by interpolating the function $Y(x)$ given by:

$$Y(x) = \sum_{j=0}^{r+s-1} \varphi_j x^j \tag{2.3}$$

where,

i) $x \in [a, b]$

ii) φ_j are unknowns coefficients to be determined.

iii) r is the number of interpolations for $1 \le r \le k$ and

iv) s is the number of distinct collocation points with $s > 0$

v) q is a chosen off-grid interpolation point.

The collocation approximation is constructed by imposing the following conditions:

$$Y(x_{n+j}) = y_{n+j},\ j = 0(1)2,...,r-1 \quad (2.4)$$

$$Y''(x_{n+\mu}) = y_{n+\mu},\ \mu = \{j, v_1, v_2\},\ j = 0,1,...,k \quad (2.5)$$

where v_1 and v_2 are not integers.

Interpolating (2.3) at grid and off-grid point, while collocating (2.4) at some off-grid point(s) leads to a system of equations which can be put into matrix form:

$$DC = I \quad (2.6)$$

where,

I is the identity matrix of dimension $(s+r)\times(s+r)$, C and D being square matrices of dimension $(s+r)\times(s+r)$ each. Matrix D is inverted to obtain the columns of $C = D^{-1}$ which gives the continuous coefficients $\alpha_j(x), \alpha_q(x)$ and $\beta_j(x)$ in (2.2).

Derivation of Two-Step Block Hybrid Numerov Method (*BhyNM* 1): with one off-grid interpolation point.

We consider $r = 3, s = 3$ and $q = \frac{1}{2}$ and equation (2.2) yields

$$y(x) = \sum_{j=0}^{2}\alpha_j(x)y_{n+j} + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h^2\sum_{j=0}^{2}\beta_j(x)f_{n+j} \quad (3.1)$$

Equation (3.1) can be put into matrix form (2.6) where D is obtained as:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \end{bmatrix} \quad (3.2)$$

and the following schemes are derived from (3.2) namely:
Main Method:

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}[f_{n+2} + 10f_{n+1} + f_n]$$

Additional Method:

$$\frac{24}{5}y_{n+1} - \frac{48}{5}y_{n+\frac{1}{2}} + \frac{24}{5}y_n = \frac{h^2}{10}[f_{n+1} + 10f_{n+\frac{1}{2}} + f_n]$$

$$\frac{53}{75}y_{n+1} - \frac{256}{75}y_{n+\frac{1}{2}} + \frac{203}{75}y_n = -hy'_n + \frac{h^2}{3600}[-10f_{n+2} + 188f_{n+1} - 442f_n] \quad (3.3)$$

and equations of the first derivative given by:

$$\frac{83}{75}y_{n+1} - \frac{14}{75}y_{n+\frac{1}{2}} - \frac{68}{75}y_n = -hy'_{n+\frac{1}{2}} + \frac{h^2}{14400}[-35f_{n+2} + 838f_{n+1} - 467f_n]$$

$$\frac{187}{75}y_{n+1} - \frac{224}{75}y_{n+\frac{1}{2}} + \frac{37}{75}y_n = -hy'_{n+1} + \frac{h^2}{1800}[5f_{n+2} - 274f_{n+1} + 41f_n]$$

$$\frac{53}{75}y_{n+1} - \frac{256}{75}y_{n+\frac{1}{2}} + \frac{203}{75}y_n = -hy'_{n+2} + \frac{h^2}{1800}[595f_{n+2} + 2494f_{n+1} + 379f_n] \quad (3.4)$$

Derivation of Two-Step Block Hybrid Numerov Method (*BhyNM 2*): with two off-grid interpolation points.

We consider $r = 4, s = 3$ and $q = \frac{1}{2}, \frac{3}{2}$ where matrix D is obtained as

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+\frac{3}{2}} & x^2_{n+\frac{3}{2}} & x^3_{n+\frac{3}{2}} & x^4_{n+\frac{3}{2}} & x^5_{n+\frac{3}{2}} & x^6_{n+\frac{3}{2}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \end{bmatrix} \quad (4.1)$$

and the following equations are obtained:
Main Method:

$$y_{n+2} - \frac{32}{3}y_{n+\frac{3}{2}} + \frac{58}{3}y_{n+1} - \frac{32}{3}y_{n+\frac{1}{2}} + y_n = \frac{h^2}{36}[f_{n+2} - 62f_{n+1} + f_n]$$

Additional Method:

$$\frac{4}{5}y_{n+\frac{3}{2}} - \frac{32}{5}y_{n+1} + \frac{52}{5}y_{n+\frac{1}{2}} - \frac{24}{5}y_n = \frac{h^2}{240}[f_{n+2} + 22f_{n+1} - 240f_{n+\frac{1}{2}} - 23f_n]$$

$$-\frac{204}{5}y_{n+\frac{3}{2}} + \frac{432}{5}y_{n+1} - \frac{252}{5}y_{n+\frac{1}{2}} + \frac{24}{5}y_n = \frac{h^2}{80}[3f_{n+2} - 80f_{n+\frac{3}{2}} - 654f_{n+1} + 11f_n]$$

$$\frac{11}{75}y_{n+\frac{3}{2}} - \frac{57}{25}y_{n+1} - \frac{48}{25}y_{n+\frac{1}{2}} + \frac{203}{75}y_n = hy'_n + \frac{h^2}{200}[f_{n+2} + 82f_{n+1} - 23f_n] \quad (4.2)$$

Evaluating 1st derivative at some grid and off-grid points yield the following equations:

Science World Journal Vol 12(No 4) 2017
www.scienceworldjournal.org
ISSN 1597-6343

$$\frac{259}{225}y_{n+\frac{3}{2}}-\frac{272}{225}y_{n+1}+\frac{217}{225}y_{n+\frac{1}{2}}-\frac{68}{75}y_n=hy'_{n+\frac{1}{2}}+\frac{h^2}{21600}[77f_{n+2}+7214f_{n+1}-571f_n]$$

$$\frac{256}{225}y_{n+\frac{3}{2}}-\frac{1073}{225}y_{n+1}+\frac{928}{225}y_{n+\frac{1}{2}}-\frac{37}{75}y_n=hy'_{n+1}+\frac{h^2}{5400}[17f_{n+2}+2294f_{n+1}-91f_n]$$

$$\frac{653}{75}y_{n+\frac{3}{2}}-\frac{408}{25}y_{n+1}+\frac{213}{25}y_{n+\frac{1}{2}}-\frac{68}{75}y_n=hy'_{n+\frac{3}{2}}+\frac{h^2}{800}[f_{n+2}+982f_{n+1}-23f_n]$$

$$\frac{6064}{225}y_{n+\frac{3}{2}}-\frac{12287}{225}y_{n+1}+\frac{6832}{225}y_{n+\frac{1}{2}}-\frac{203}{75}y_n=hy'_{n+2}+\frac{h^2}{5400}[-1027f_{n+2}+27386f_{n+1}-379f_n] \quad (4.3)$$

Derivation of Two-Step Block Hybrid Numerov Method (*BhyNM 3*): with three off-grid interpolation points.

We consider $r=5, s=3$ with $q=\frac{1}{2},\frac{3}{4}$ and $\frac{3}{2}$. Then D matrix is obtained as:

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\
1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 \\
1 & x_{n+\frac{3}{4}} & x_{n+\frac{3}{4}}^2 & x_{n+\frac{3}{4}}^3 & x_{n+\frac{3}{4}}^4 & x_{n+\frac{3}{4}}^5 & x_{n+\frac{3}{4}}^6 & x_{n+\frac{3}{4}}^7 \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\
1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5
\end{bmatrix} \quad (5.1)$$

And the following equations are obtained:
Main Method:

$$y_{n+2}-\frac{32}{3}y_{n+\frac{3}{2}}+\frac{58}{3}y_{n+1}-\frac{32}{3}y_{n+\frac{1}{2}}+y_n=\frac{h^2}{36}[f_{n+2}-62f_{n+1}+f_n]$$

Additional Method:

$$\frac{116}{81}y_{n+\frac{3}{2}}+\frac{64}{9}y_{n+1}+\frac{1024}{81}y_{n+\frac{3}{4}}+\frac{4}{9}y_{n+\frac{1}{2}}+\frac{296}{81}y_n=\frac{h^2}{432}[f_{n+2}+262f_{n+1}+432f_{n+\frac{1}{2}}+25f_n]$$

$$\frac{3124}{81}y_{n+\frac{3}{2}}-\frac{784}{9}y_{n+1}+\frac{1024}{81}y_{n+\frac{3}{4}}+\frac{356}{9}y_{n+\frac{1}{2}}-\frac{296}{81}y_n=\frac{h^2}{432}[-19f_{n+2}+432f_{n+\frac{3}{2}}+3230f_{n+1}-43f_n]$$

$$\frac{10}{81}y_{n+\frac{3}{2}}-\frac{335}{18}y_{n+1}+\frac{3040}{81}y_{n+\frac{3}{4}}-\frac{175}{9}y_{n+\frac{1}{2}}+\frac{65}{162}y_n=\frac{h^2}{3456}[f_{n+2}-170f_{n+1}-3456f_{n+\frac{3}{4}}+25f_n]$$

$$\frac{1552}{2835}y_{n+\frac{3}{2}}+\frac{923}{315}y_{n+1}-\frac{32768}{2835}y_{n+\frac{3}{4}}+\frac{3728}{315}y_{n+\frac{1}{2}}-\frac{10643}{2835}y_n=hy'_n+\frac{h^2}{7560}[7f_{n+2}+1726f_{n+1}+607f_n] \quad (5.2)$$

Equations of the first derivative given by:

$$\frac{443}{2835}y_{n+\frac{3}{2}}+\frac{512}{315}y_{n+1}-\frac{20992}{2835}y_{n+\frac{3}{4}}+\frac{1697}{315}y_{n+\frac{1}{2}}+\frac{668}{2835}y_n=-hy'_{n+\frac{1}{2}}+\frac{h^2}{30240}[7f_{n+2}+2266f_{n+1}+127f_n]$$

$$\frac{2272}{25515}y_{n+\frac{3}{2}}-\frac{14467}{2835}y_{n+1}+\frac{151552}{25515}y_{n+\frac{3}{4}}-\frac{2752}{2835}y_{n+\frac{1}{2}}+\frac{1147}{25515}y_n=-hy'_{n+1}+\frac{h^2}{68040}[7f_{n+2}+6586f_{n+1}+67f_n]$$

$$\frac{20977}{2835}y_{n+\frac{3}{2}}-\frac{5272}{315}y_{n+1}+\frac{20992}{2835}y_{n+\frac{3}{4}}+\frac{683}{315}y_{n+\frac{1}{2}}-\frac{668}{2835}y_n=hy'_{n+\frac{3}{2}}+\frac{h^2}{30240}[-77f_{n+2}+24754f_{n+1}-197f_n]$$

$$\frac{7769}{90720}y_{n+\frac{3}{2}}+\frac{10673}{5040}y_{n+1}-\frac{1268}{2835}y_{n+\frac{3}{4}}-\frac{18359}{10080}y_{n+\frac{1}{2}}+\frac{1481}{22680}y_n=hy'_{n+\frac{3}{4}}+\frac{h^2}{1935360}[217f_{n+2}+97246f_{n+1}+2617f_n]$$

$$-\frac{82192}{2835}y_{n+\frac{3}{2}}+\frac{16997}{315}y_{n+1}-\frac{32768}{2835}y_{n+\frac{3}{4}}-\frac{12688}{315}y_{n+\frac{1}{2}}+\frac{10643}{2835}y_n=hy'_{n+2}+\frac{h^2}{7560}[1393f_{n+2}-43166f_{n+1}+793f_n] \quad (5.3)$$

Order, Consistency, Zero-Stability of the Methods

Extending the idea of Henrici [4] and Jator [6], the linear difference operator L associated with (2.2) is defined by

$$L[y(x);h]=\sum_{j=0}^{k}[\alpha_j y(x+jh)-h^2\beta_j y''(x+jh)] \quad (6.1)$$

where, $y(x)$ is an arbitrary function continuously differentiable on the interval $[a,b]$. We expand the test function $y(x+jh)$ and its second derivative $y''(x+jh)$ about x and collect the terms to obtain

$$L[y(x);h]=c_0y(x)+c_1hy'(x)+\ldots+c_qh^qy^q(x)+\ldots$$

$$c_0=\alpha_0+\alpha_1+\ldots+\alpha_k$$

$$c_1=\alpha_1+2\alpha_2+\ldots+k\alpha_k$$

$$c_2=\frac{1}{2!}\left(\alpha_1+2^2\alpha_2+\ldots+k^2\alpha_k\right)-\left(\beta_1+\beta_2+\ldots+\beta_k\right)$$

$$\cdot c_q=\frac{1}{q!}\left(\alpha_1+2^q\alpha_2+\ldots+k^q\alpha_k\right)-\frac{1}{(q-2)!}\left(\beta_1+2^{q-2}\beta_2+\ldots+k^{q-2}\beta_k\right),$$

$$q=2,3,\ldots \quad (6.2)$$

Definition 6.1
The method (2.2) is said to be of order p if $c_0=c_1=\ldots=c_{p+1}=0$ and $c_{p+2}\neq 0$ where c_{p+2} is called the error constant, $c_{p+2}h^{p+2}y^{(p+2)}(x_n)$ is the truncation error at point x_n.

Definition 6.2
A linear multistep method (2.2) is consistent if it has order $p\geq 1$.

Definition 6.3
The block method of the form (3.3), (4.2) and (5.2) is said to be zero stable if as $h\to 0$, the roots $\lambda_j, j=1(2)k$ of the first characteristic polynomials $\rho(\lambda)$ is given by

$$\rho(\lambda)=\det\left[\sum_{i=0}^{k}A^i\lambda^{k-1}\right]=0 \quad (6.3)$$

Satisfies $|\lambda|\leq 1$ the multiplicity must not exceed two, Fatunla [2].

Definition 6.4
A linear multistep method (2.2) is convergent iff it is consistent and zero-stable.
As a consequence to definitions (6.1) and (6.2), the block hybrid method of the form (3.3), (4.2) and (5.2) respectively is consistent.

Zero Stability of *BhyNM 1*

To analyze the zero stability, the block hybrid method (3.3) is normalized. Zero stability can be described by matrix finite difference equation as follows:

$$I_3 Y_{m+1} = A^{(1)} Y_m + h^2 [B^{(0)} F_{m+1} + B^{(1)} F_m] \qquad (6.4)$$

with

$Y_{m+1} = [y_{n+\frac{1}{2}}, y_{n+1}, y_{n+2}]^T$, $Y_m = [y_{n-1}, y_{n-\frac{1}{2}}, y_n]^T$,

$F_{m+1} = [f_{n+\frac{1}{2}}, f_{n+1}, f_{n+2}]^T$ and

$F_m = [f_{n-1}, f_{n-\frac{1}{2}}, f_n]^T$

and constant matrices

$I_3 = 3\times 3$ identity matrix.

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} \frac{53}{720} & -\frac{3}{160} & \frac{1}{720} \\ \frac{16}{45} & -\frac{1}{60} & \frac{1}{360} \\ \frac{32}{45} & \frac{4}{5} & \frac{4}{45} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} 0 & 0 & \frac{11}{160} \\ 0 & 0 & \frac{19}{120} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

The first characteristic polynomial is define as

$$\rho(\lambda) = \det\left[\lambda A^{(0)} - A^{(1)}\right] = 0$$

$$= \lambda^2(\lambda - 1) = 0$$

Since the roots of the first characteristic polynomial has modulus less than or equal to one, the block method (3.3) is zero stable. Hence, by definition (6.4), the method is convergent.

Zero Stability of *BhyNM 2*

The block hybrid method (4.2) can be described equation (6.4) where,

$Y_{m+1} = [y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}]^T$,

$Y_m = [y_{n-\frac{3}{2}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n]^T$,

$F_{m+1} = [f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}]^T$ and

$F_m = [f_{n-\frac{3}{2}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n]^T$

and $I_4 = 4\times 4$ identity matrix.

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} \frac{3}{32} & -\frac{47}{960} & \frac{29}{1440} & -\frac{7}{1920} \\ \frac{2}{5} & -\frac{1}{12} & \frac{2}{45} & -\frac{1}{120} \\ \frac{117}{160} & \frac{27}{320} & \frac{3}{32} & -\frac{9}{640} \\ \frac{16}{15} & \frac{4}{15} & \frac{16}{45} & 0 \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \frac{367}{5760} \\ 0 & 0 & 0 & \frac{53}{360} \\ 0 & 0 & 0 & \frac{147}{640} \\ 0 & 0 & 0 & \frac{14}{45} \end{bmatrix}$$

and equation (6.3) becomes

$$\rho(\lambda) = \det\left[\lambda A^{(0)} - A^{(1)}\right] = 0$$

$$= \lambda^3(\lambda - 1) = 0 \;\Rightarrow \lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 0$$

We have therefore shown that the block hybrid method (4.2) is zero-stable. Following definition (6.4), the method is convergent.

Zero Stability of *BhyNM 3*

The block hybrid method (5.2) can be described equation (6.4) where, $I_5 = 5\times 5$ identity matrix.

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B^{(0)} = \begin{bmatrix} \frac{403}{2016} & -\frac{214}{945} & \frac{247}{2240} & -\frac{461}{30240} & \frac{67}{40320} \\ \frac{15201}{35840} & -\frac{459}{1120} & \frac{7209}{35840} & -\frac{993}{35840} & \frac{27}{8960} \\ \frac{206}{315} & -\frac{512}{945} & \frac{25}{84} & -\frac{38}{945} & \frac{11}{2520} \\ \frac{1269}{1120} & -\frac{6}{7} & \frac{1539}{2240} & -\frac{9}{224} & \frac{27}{4480} \\ \frac{496}{315} & -\frac{1024}{945} & \frac{36}{35} & \frac{176}{945} & \frac{8}{315} \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{6637}{120960} \\ 0 & 0 & 0 & 0 & \frac{3243}{35840} \\ 0 & 0 & 0 & 0 & \frac{953}{7560} \\ 0 & 0 & 0 & 0 & \frac{879}{4480} \\ 0 & 0 & 0 & 0 & \frac{254}{945} \end{bmatrix}$$

then equation (6.3) yields:

$$\rho(\lambda) = \det\left[\lambda A^{(0)} - A^{(1)}\right] = 0$$

$$= \lambda^4(\lambda - 1) = 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$$

By definition (6.3), the block method (5.2) is zero stable. Hence, by definition (6.4), the method is also convergent.

1. Numerical Examples and Discussion.

To illustrate the effectiveness of the block hybrid Numerov type method (*BhyNM* 1, *BhyNM* 2 and *BhyNM* 3) for step size $h = \frac{1}{100}$, the following test examples are solved numerically.

For sake of implementation, we combine equations (3.3) and their first derivative equations (3.4), we also combine (4.2) with (4.3) as well as combine (5.2) with (5.3). This procedure leads to a system of six, eight and ten equations each to be solved simultaneously to give us the approximate solutions $y_i, i = 1,2,\ldots$, this approach has been proven to be a good self-starting procedure for the solution of (1.1) without reduction to first order ODE.

Example 1

Consider the second order ODE

$$y'' = 4y' - 4y,\ y(0) = 12,\ y'(0) = -3,\ 0 \le x \le 1$$

The problem is known to have theoretical solution

$$y(x) = 12e^{2x} - 27xe^{2x}$$

Example 2

$$y'' = 8y' - 17y,\ y(0) = -4,\ y'(0) = -1,\ 0 \le x \le 1$$

The problem is known to have theoretical solution

$$y(x) = -4e^{4x}\cos(x) + 15e^{4x}\sin(x)$$

RESULTS

Table 1: Numerical Solution for example 1

x	*Theoretical Solution*	*BhyNM 1*	*BhyNM 2*	*BhyNM 3*
0.00	12.0000000000000	12.0000000000000	12.0000000000000	12.0000000000000
0.01	11.9669617185138	11.9669617185106	11.9669617185139	11.9669617185117
0.02	11.9276914722448	11.9276914722193	11.9276914722446	11.9276914722401
0.03	11.8819509558426	11.8819509557617	11.8819509558424	11.8819509558355
0.04	11.8294947790105	11.8294947788522	11.8294947790101	11.8294947790009
0.05	11.7700702775056	11.7700702772329	11.7700702775051	11.7700702774934
0.06	11.7034173193939	11.7034173189814	11.7034173193931	11.7034173193788
0.07	11.6292681064466	11.6292681058532	11.6292681064456	11.6292681064286
0.08	11.5473469705594	11.5473469697563	11.5473469705581	11.5473469705383
0.09	11.4573701650757	11.4573701640174	11.4573701650742	11.4573701650515
0.10	11.3590456508896	11.3590456495431	11.3590456508876	11.3590456508620

Table 2: Maximum Errors for example 1

x	*BhyNM 1* (E_{max})	*BhyNM 2* (E_{max})	*BhyNM 3* (E_{max})
0.00	0.000e+00	0.000e+00	0.000e+00
0.01	3.200e-12	1.000e-13	2.100e-12
0.02	2.550e-11	2.000e-13	4.700e-12
0.03	8.090e-11	2.000e-13	7.100e-12
0.04	1.583e-10	4.000e-13	9.600e-12
0.05	2.727e-10	5.000e-13	1.220e-11
0.06	4.125e-10	8.000e-13	1.510e-11
0.07	5.934e-10	1.000e-12	1.800e-11
0.08	8.031e-10	1.300e-12	2.110e-11
0.09	1.058e-09	1.500e-12	2.420e-11
0.10	1.347e-09	2.000e-12	2.760e-11

Science World Journal Vol 12(No 4) 2017
www.scienceworldjournal.org
ISSN 1597-6343

Table 3: Numerical Solution for example 2

x	Theoretical Solution	BhyNM 1	BhyNM 2	BhyNM 3
0.00	-4.00000000000000	-4.00000000000000	-4.00000000000000	-4.00000000000000
0.01	-4.00691592223446	-4.00691592218072	-4.00691592223516	-4.0069159223366
0.02	-4.00731721493852	-4.00731721451289	-4.00731721494002	-4.00731721493683
0.03	-4.00066058359423	-4.00066058222170	-4.00066058359662	-4.00066058359166
0.04	-3.98636997397524	-3.98636997123705	-3.98636997397863	-3.98636997397170
0.05	-3.96383486307199	-3.96383485826489	-3.96383486307649	-3.96383486306748
0.06	-3.93240846866907	-3.93240846125594	-3.93240846867480	-3.93240846866349
0.07	-3.89140587396774	-3.89140586309653	-3.89140587397483	-3.89140587396111
0.08	-3.84010206349679	-3.84010204849180	-3.84010206350538	-3.84010206348901
0.09	-3.77772986639864	-3.77772984623567	-3.77772986640887	-3.77772986638971
0.10	-3.70347780301604	-3.70347777685782	-3.70347780302807	-3.70347780300587

Table 4: Maximum Errors for example 2

x	BhyNM 1 (E_{max})	BhyNM 2 (E_{max})	BhyNM 3 (E_{max})
0.00	0.000e+00	0.000e+00	0.000e+00
0.01	5.374e-11	7.000e-13	8.000e-13
0.02	4.256e-10	1.500e-12	1.690e-12
0.03	1.373e-9	2.390e-12	2.570e-12
0.04	2.738e-9	3.390e-12	3.540e-12
0.05	4.807e-9	4.500e-12	4.510e-12
0.06	7.413e-9	5.730e-12	5.580e-12
0.07	1.087e-8	7.090e-12	6.630e-12
0.08	1.501e-8	8.590e-12	7.780e-12
0.09	2.016e-8	1.023e-11	8.930e-12
0.10	2.616e-8	1.203e-11	1.017e-11

$$E_{\max} = \max\left[\left|y(x_i)_{calculated} - y(x_i)_{theoretical}\right|\right], i = 1,2,...$$

Table 5: Order and Error Constants of *BhyNM* 1

$Equation(s)$	p	c_{p+2}
$3.3a$	4	$-\frac{1}{240}$
$3.3b$	4	$-\frac{1}{3200}$
$3.3c$	4	$\frac{1}{12700}$

Table 6: Equations of 1st Derivative of (3.2)

$Equation(s)$	p	c_{p+2}
$3.4a$	4	$-\frac{259}{576000}$
$3.4b$	4	$-\frac{1}{2250}$
$3.4c$	4	$-\frac{379}{36000}$

Table 7: Order and Error Constants of *BhyNM* 2

$Equation(s)$	p	c_{p+2}
$4.2a$	6	$-\frac{43}{1451520}$
$4.2b$	5	$-\frac{1}{6400}$
$4.2c$	5	$\frac{1}{6400}$
$4.2d$	5	$-\frac{1}{7000}$

Table 8: Equations of 1st Derivative of (4.1)

$Equation(s)$	p	c_{p+2}
$4.3a$	5	$-\frac{41}{448000}$
$4.3b$	5	$-\frac{37}{504000}$
$4.3c$	5	$-\frac{41}{448000}$
$4.3d$	5	$\frac{1}{7000}$

Table 9: Order and Error Constants of *BhyNM* 3

$Equation(s)$	p	c_{p+2}
$5.2a$	6	$-\frac{43}{1451520}$
$5.2b$	6	$-\frac{1009}{69672960}$
$5.2c$	6	$\frac{6589}{69672960}$
$5.2d$	6	$-\frac{899}{445906944}$
$5.2e$	6	$-\frac{1921}{304819200}$

Table 10: Equations of 1st Derivative of (5.1)

$Equation(s)$	p	c_{p+2}
$5.3a$	6	$-\frac{6469}{4877107200}$
$5.3b$	6	$-\frac{2627}{5486745600}$
$5.3c$	6	$\frac{29849}{4877107200}$
$5.3d$	6	$-\frac{356123}{624269721600}$
$5.3e$	6	$-\frac{911209}{7620480}$

DISCUSSION

We developed some generalized block Numerov method of hybrid type (*BhyNM* 1, *BhyNM* 2 and *BhyNM* 3) via multistep collocation technique for a direct numerical solution to the general second order ODE of the form (1.1). The methods developed proved to be very efficient in terms of accuracy and speed (see, tables 1 and 3) due to their order and high error constants (refer to, tables 5-10). Their absolute maximum errors (E_{max}) suggest that they converge much faster to the analytical solution than other methods of the same order (see, tables 2 and 4). The regions of absolute stability of our block hybrid method were also presented (Fig. 1, 2 and 3)

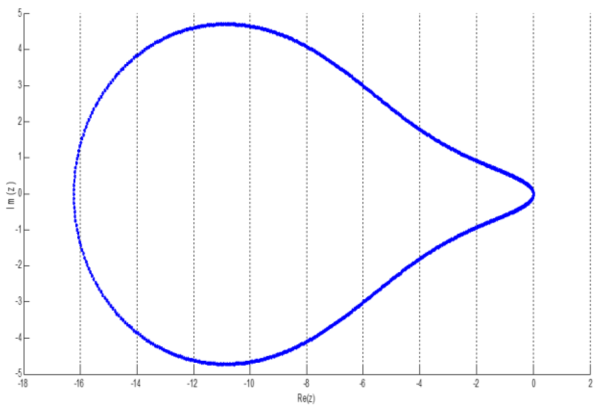

Fig.1: Region of absolute stability for *BhyNM* 1

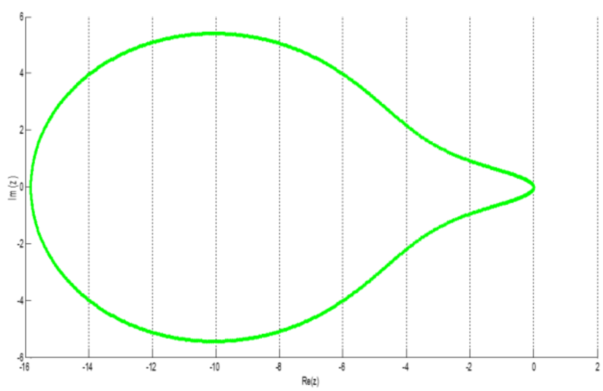

Fig.2: Region of absolute stability for *BhyNM 2*

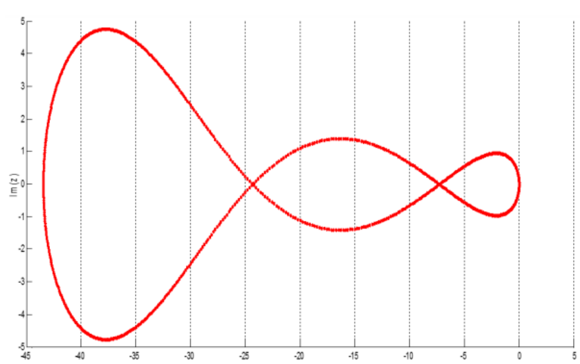

Fig.3: Region of absolute stability for *BhyNM* 3

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