THE CONCEPT OF $\alpha$-CUTS IN MULTI $Q$-FUZZY SET

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ABSTRACT
The purpose of this paper is to introduce the concept of $\alpha$-Cuts and their properties in multi $Q$-fuzzy sets. In addition, both first and second decomposition theorems were established and proved. It is shown that any Multi $Q$-fuzzy Set can be represented as the union of its special $\alpha$-cuts as well as its special strong $\alpha$-cuts.

Keywords: Fuzzy multiset, multi $Q$-fuzzy set, $\alpha$-Cut.

1. INTRODUCTION
A fuzzy set which is a generalized set of objects occurring with a continuum of degrees of membership was introduced by (Zadeh, 1965); he further showed the application of $\alpha$-cuts to fuzzy sets. For the basics of fuzzy set and its applications refer to (Brown, 1971; Singh et al., 2015; Goguen, 1967; Wygralak, 1989; Chutia et al., 2010; Dutta et al., 2011; Klir and Yuan, 1995; Kreinovich, 2013). Singh et al. (2014) studied $\alpha$-cuts and some of its properties in fuzzy multisets. Multi $Q$-fuzzy set was studied in various contexts in (Adam and Hassan, 2014a; Adam and Hassan, 2014b; Adam and Hassan, 2015; Adam and Hassan, 2016); their relevant applications were shown. In this paper, $\alpha$-cuts and their properties in Multi $Q$-fuzzy sets were studied.

2. Preliminaries

Definition 2.1 Multiset
An mset A drawn from the set X is represented by a function Count $A$ or $C_{A}$ defined as $C_{A}$: $X \rightarrow \mathbb{N}$. One way of representing a multiset A from X with $x_{1}$ appearing $k_{1}$ times, $x_{2}$ appearing $k_{2}$ times etc., is $A = \{k_{1}x_{1}, k_{2}x_{2}, ..., k_{n}x_{n}\}$, where $x_{i} \in X$.

Let A and B be two multisets drawn from a set X. Then
\[ A \subseteq B \iff C_{A}(x) \leq C_{B}(x) \quad \text{for all } x \in X. \]
\[ A = B \iff C_{A}(x) = C_{B}(x) \quad \text{for all } x \in X. \]
\[ A \cup B = \max(C_{A}(x), C_{B}(x)) \quad \text{for all } x \in X. \]
\[ A \cap B = \min(C_{A}(x), C_{B}(x)) \quad \text{for all } x \in X. \]

(Jena et al., 2001)

Definition 2.2 Fuzzy Multiset
A fuzzy multiset A is a multiset of pairs, where the first part of each pair is an element of a universe set X and the second part is the degree to which the first part belongs to that fuzzy multiset. That is, $A: X \times I \rightarrow \mathbb{N}$; where $I = [0,1]$ and $\mathbb{N}$ is the set of positive integers including 0 (Syropoulos, 2012).

Let A and B be fuzzy multisets. Then, Lengths $L(x; A)$ and $L(x; A, B)$ are respectively defined as
\[ L(x; A) = \max\{j; \mu_{A}(x) \neq 0\} ; \quad \text{and} \]
\[ L(x; A, B) = \max(L(x; A), L(x; B)) . \]

For brevity, $L(x)$ for $L(x; A)$ or $L(x; A, B)$ is also used if no confusion arises.

Note that for defining an operation between two fuzzy multisets A and B, the lengths of the membership sequences $\mu_{A}(x), \mu_{B}(x), ...$ need to be set equal.

Let A, B be fuzzy multisets. Then
\[ A \cup B = \mu_{A \cup B}(x) = \mu_{A}(x) \lor \mu_{B}(x), j = 1, ..., L(x), \forall x \in X. \]
\[ A \cap B = \mu_{A \cap B}(x) = \mu_{A}(x) \land \mu_{B}(x), j = 1, ..., L(x), \forall x \in X. \]
\[ A \subseteq B \iff \mu_{A}(x) \leq \mu_{B}(x), j = 1, ..., L(x), \forall x \in X. \]
\[ A = B \iff A \subseteq B \quad \text{and} \quad B \subseteq A \]

Definition 2.3 Multi $Q$-fuzzy Set
Let $J$ be a unit interval $[0,1]$, $k$ be a positive integer, $U$ be a universal set and $Q$ be a non-empty set. A multi $Q$-fuzzy set $A_{Q}$ in $U$ and $Q$ is a set of ordered sequences:
\[ A_{Q} = \{(u, q_{1}), (u, q_{2}), (u, q_{3}), ..., (u, q_{k})\}: u \in U, q \in Q \}, \]
where $\mu_{u}(u) \in I$ for all $i = 1, 2, ..., k$.

The function $(u_{1}(u, q), u_{2}(u, q), ..., u_{k}(u, q))$ is called the membership function of multi $Q$-fuzzy set $A_{Q}$ and $u_{1}(u, q) + u_{2}(u, q) + \cdots + u_{k}(u, q) \leq 1, k$ is called the dimension of $A_{Q}$ (Adam and Hassan, 2014a).

In other words, if the sequences of the membership functions have only $k$ terms (finite number of terms) the multi $Q$-fuzzy set is a function from $U \times Q$ to $I^{k}$ such that for all $(u, q) \in U \times Q$, $\mu_{u}(u, q) = (u_{1}(u, q), u_{2}(u, q), ..., u_{k}(u, q))$. The set of all multi $Q$-fuzzy sets of dimension $k$ in $U$ and $Q$ is denoted by $M^{k}QF(U)$.

3. The Concept of $\alpha$-cuts in Multi $Q$-fuzzy set

Definition 3.1 $\alpha$-cuts in multi $Q$-fuzzy set
Let $A_{Q} \in M^{k}QF(U)$ and $\alpha \in [0,1]$. Then the $\alpha$ -cut of $A_{Q}$, denoted $\alpha^{-}A_{Q}$ is defined as
\[ \alpha^{-}A_{Q} = \{(u, q): \mu_{u}(u, q) \geq \alpha\}. \]

The strong $\alpha^{-}$ cut of $A_{Q}$, denoted $\alpha^{+}A_{Q}$ is defined as
\[ \alpha^{+}A_{Q} = \{(u, q): \mu_{u}(u, q) > \alpha\}. \]

Theorem 3.2 Let $A_{Q}, B_{Q} \in M^{k}QF(U)$ and $\alpha \in [0,1]$. Then
\[ \alpha^{-}A_{Q} \subseteq \alpha^{-}A_{Q} \]
\[ \alpha^{-}A_{Q} \cup \alpha^{-}B_{Q} = \alpha^{-}A_{Q} \cup \alpha^{-}B_{Q} \]
\[ \alpha^{-}A_{Q} \cap \alpha^{-}B_{Q} = \alpha^{-}A_{Q} \cap \alpha^{-}B_{Q} \]
\[ \alpha^{+}A_{Q} \subseteq \alpha^{+}A_{Q} \]
\[ \alpha^{+}A_{Q} \cup \alpha^{+}B_{Q} = \alpha^{+}A_{Q} \cup \alpha^{+}B_{Q} \]
\[ \alpha^{+}A_{Q} \cap \alpha^{+}B_{Q} = \alpha^{+}A_{Q} \cap \alpha^{+}B_{Q} \]
Proof

i). Observe from definition 3.1; \( \alpha - \text{cut} \) always contains strong \( \alpha - \text{cut} \).

ii). From definition 3.1; observe that whenever \( \alpha_1 \leq \alpha_2 \) automatically \( A_{\alpha_1} \) will contain \( A_{\alpha_2} \).

iii). Let \( (u, q) \in \alpha^*(A_{\alpha} \cup B_{\alpha}) \Rightarrow \mu^i_{\alpha} (A_{\alpha} \cup B_{\alpha}) (u, q) \geq \alpha, i = 1, 2, ..., k \)

\[ \Rightarrow \max \left[ \mu^i_{A_{\alpha}} (u, q), \mu^i_{B_{\alpha}} (u, q) \right] \geq \alpha, i = 1, 2, ..., k \]

\[ \Rightarrow \mu^i_{A_{\alpha}} (u, q) \geq \alpha \text{ or } \mu^i_{B_{\alpha}} (u, q) \geq \alpha, i = 1, 2, ..., k \]

\[ \Rightarrow (u, q) \in A_{\alpha} \text{ or } (u, q) \in B_{\alpha} \]

\[ \Rightarrow (u, q) \in a_{\alpha} \cup a_{B_{\alpha}} \]

\[ \Rightarrow a_{\alpha} \cup a_{B_{\alpha}} \subseteq a_{\alpha} \cup a_{B_{\alpha}}. \]

Suppose \( (u, q) \in a_{\alpha} \cup a_{B_{\alpha}} \)

\[ \Rightarrow (u, q) \in A_{\alpha} \text{ or } (u, q) \in B_{\alpha} \]

\[ \Rightarrow \mu^i_{A_{\alpha}} (u, q) \geq \alpha \text{ or } \mu^i_{B_{\alpha}} (u, q) \geq \alpha, i = 1, 2, ..., k \]

\[ \Rightarrow \mu^i_{(A_{\alpha} \cup B_{\alpha})} (u, q) \geq \alpha, i = 1, 2, ..., k \]

\[ \Rightarrow (u, q) \in A_{\alpha} \cup B_{\alpha} \]

\[ \Rightarrow A_{\alpha} \cup B_{\alpha} \subseteq a_{\alpha} \cup a_{B_{\alpha}}. \]

Thus, the result follows.

iv). The proof follows similarly from (iii).

v). The proof follows similarly from (iv).

vi). Let \( (u, q) \in a_{\alpha}^* (A_{\alpha} \cap B_{\alpha}) \Rightarrow \mu^i_{(A_{\alpha} \cap B_{\alpha})} (u, q) \geq \alpha, i = 1, 2, ..., k \)

\[ \Rightarrow \min \left[ \mu^i_{A_{\alpha}} (u, q), \mu^i_{B_{\alpha}} (u, q) \right] \geq \alpha, i = 1, 2, ..., k \]

\[ \Rightarrow \mu^i_{A_{\alpha}} (u, q) \geq \alpha, i = 1, 2, ..., k \] and

\[ \mu^i_{B_{\alpha}} (u, q) \geq \alpha, i = 1, 2, ..., k \]

\[ \Rightarrow (u, q) \in a_{\alpha} \cap B_{\alpha} \text{ and } (u, q) \in a_{\alpha} \cap B_{\alpha} \]

\[ \Rightarrow (u, q) \in a_{\alpha} \cap a_{B_{\alpha}} \]

\[ \Rightarrow a_{\alpha} \cap a_{B_{\alpha}} \subseteq a_{\alpha} \cap a_{B_{\alpha}}. \]

Also, let \( (u, q) \in a_{\alpha} \cap a_{B_{\alpha}} \)

\[ \Rightarrow (u, q) \in a_{\alpha} \cap a_{B_{\alpha}} \]

\[ \Rightarrow \mu^i_{A_{\alpha} \cap B_{\alpha}} (u, q) \geq \alpha \text{ and } i = 1, 2, ..., k \]

\[ \Rightarrow \mu^i_{(A_{\alpha} \cap B_{\alpha})} (u, q) \geq \alpha, i = 1, 2, ..., k \]

\[ \Rightarrow (u, q) \in a_{\alpha} \cap B_{\alpha} \]

\[ \Rightarrow a_{\alpha} \cap B_{\alpha} \subseteq a_{\alpha} \cap a_{B_{\alpha}}. \]

Hence, \( a_{\alpha} \cup a_{B_{\alpha}} = a_{\alpha} \cap a_{B_{\alpha}}. \)

Definition 3.3 Decomposition of Multi Q-fuzzy Soft Set

Let \( U = \{u_1, u_2, ..., u_n\} \) and \( Q = \{q_1, q_2, ..., q_n\} \) over \( U \) and \( Q \) be a Multi Q-fuzzy set \( A_{\alpha} \) over \( U \) and \( Q \) be \( A_{\alpha} = \{(u_1, p_1), (u_1, q_1), (u_2, q_2), (u_3, q_3), (u_4, q_4), (u_5, q_5)\} \).

Let have the following distinct \( \alpha \)-cuts by characteristic functions viewed as special membership functions:

\[ \alpha_{A_{\alpha}} = \{(u_1, p_1), (u_1, q_1), (u_2, q_2), (u_3, q_3), (u_4, q_4), (u_5, q_5)\}, \alpha \geq 0.1 \]

Thus, it is easy to see that \( \alpha_{A_{\alpha}} \cup \alpha_{A_{\alpha}} \cup \alpha_{A_{\alpha}} \cup \alpha_{A_{\alpha}} \cup \alpha_{A_{\alpha}} = A_{\alpha} \).

In other words, any Multi Q-fuzzy Set \( A_{\alpha} \) can be represented as the union of its special \( \alpha \)-cuts \( \alpha_{A_{\alpha}} \), and this representation is usually referred to as Decomposition of \( A_{\alpha} \).

Moreover, if each \( (u, q) \in A_{\alpha} \), \( \alpha_{A_{\alpha}} \) is defined as

\[ \alpha_{A_{\alpha}} = \alpha_{(\alpha_{A_{\alpha}})} \]

we can see that by using a similar arguments, a Multi Q-fuzzy Set \( A_{\alpha} \) can be represented as the union of its special strong \( \alpha \)-cuts \( \alpha_{A_{\alpha}} \), known as Decomposition of that Q-fuzzy Set.
Theorem 3.4 First Decomposition Theorem
Let \( A_0 \in M^k QF(U) \), then \( A_0 = \bigcup_{\alpha \in [0, 1]} \alpha A_0 \), where \( \alpha A_0 \) is as defined in (1).

Proof
For each \((u, q) \in A_0\), let \( y = \mu_{A_0}^i (u, q), i = 1, 2, ..., k \)
Then for every \( \alpha \in (y, 1] \) we have \( \mu_{A_0}^i (u, q) = y < \alpha, i = 1, 2, ..., k \). Thus, \( \alpha A_0 = 0 \).
On the other hand, for every \( \alpha \in (0, y] \) we have \( \mu_{A_0}^i (u, q) = y \geq \alpha, i = 1, 2, ..., k \).
Thus, \( \alpha A_0 = \alpha \).
Hence,
\[
(U_{\alpha \in [0, 1]} \alpha A_0) (u, q) = \sup_{\alpha \in (0, y]} \alpha = y = \\
\mu_{A_0}^i (u, q), i = 1, 2, ..., k.
\]

As the same argument is valid for each \((u, q) \in A_0\), it follows that each multi Q-fuzzy set can be uniquely represented as the family of all its \( \alpha \) -cuts.

Theorem 3.5 Second Decomposition Theorem
Let \( A_0 \in M^k QF(U) \), then \( A_0 = \bigcup_{\alpha \in [0, 1]} \alpha + A_0 \), where \( \alpha + A_0 \) is as defined in (2).

Proof
The proof is analogous to that of the above theorem.

For each \((u, q) \in A_0\), let \( y = \mu_{A_0}^i (u, q), i = 1, 2, ..., k \). Then,
\[
(U_{\alpha \in [0, 1]} \alpha + A_0) (u, q) = \sup_{\alpha \in (0, y]} \alpha + A_0 = \\
\max[ \sup_{\alpha \in (0, y]} \alpha + A_0, \sup_{\alpha \in (y, 1]} \alpha + A_0].
\]
\[
(U_{\alpha \in [0, 1]} \alpha + A_0) (u, q) = \sup_{\alpha \in (0, y]} \alpha = y = \\
\mu_{A_0}^i (u, q), i = 1, 2, ..., k.
\]

As the same argument is valid for each \((u, q) \in A_0\), it follows that each multi Q-fuzzy set can be uniquely represented as the family of all its strong \( \alpha \) -cuts.

Conclusion
The idea of \( \alpha \)-Cuts which was first applied to fuzzy set is extended to Multi Q-fuzzy set. It is shown among others that the \( \alpha \)-Cut of the union of two multi Q-fuzzy set is the same as the union of their \( \alpha \)-Cuts, and the \( \alpha \)-Cut of the intersection of two multi Q-fuzzy set is the same as the intersection of their \( \alpha \)-Cuts. It is further shown that, the same result is obtained with strong \( \alpha \)-cuts.

REFERENCES